Distance Optimal Target Assignment in Robotic Networks under Communication and Sensing Constraints

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Supported by:
The Stochastic Target Assignment Problem
The Stochastic Target Assignment Problem

\[ Q = [0,1] \times [0,1] \]
The Stochastic Target Assignment Problem

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Control: \( \dot{x}_i = u_i, \|u_i\| = 1 \)
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\( \sigma \): permutation that pairs \( x_i \) with \( y_{\sigma(i)} \)
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\[
\min_{\sigma, \{u_i\}} D_n = \sum_i \int |\dot{x}_i(t)|dt
\]
The Stochastic Target Assignment Problem, cont.
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The Stochastic Target Assignment Problem, cont.

$r_{sense}$

$r_{comm}$
The Stochastic Target Assignment Problem, cont.

\[ r_{sense} \]

\[ r_{comm} \]

\[ G(t) \]
Given $r_{sense}$ and $r_{comm}$, how can we guarantee distance optimality?
The Stochastic Target Assignment Problem, cont.

? Given $r_{sense}$ and $r_{comm}$, how can we guarantee distance optimality?
? Performance of decentralized, hierarchical strategies (algorithms)?
Related Work

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Penrose, **Random Geometric Graphs**, 2003
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Main Result

Distance optimality guarantee
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⇒ Necessary and sufficient condition for distance optimality (non-stochastic)
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Distance optimality guarantee

⇒ Necessary and sufficient condition for distance optimality (non-stochastic)
⇒ Non-asymptotic \((1 - \epsilon)\) probabilistic guarantee for \(0 < \epsilon < 1\)

\[
n \geq \begin{cases} 
\left[ \frac{\sqrt{2}}{r_{\text{sense}}} \right]^2 \log \left( \frac{1}{\epsilon} \left[ \frac{\sqrt{2}}{r_{\text{sense}}} \right]^2 \right), & r_{\text{sense}} < \frac{\sqrt{10} r_{\text{comm}}}{5} \\
\left[ \frac{\sqrt{5}}{r_{\text{comm}}} \right]^2 \log \left( \frac{1}{\epsilon} \left[ \frac{\sqrt{5}}{r_{\text{comm}}} \right]^2 \right), & r_{\text{sense}} \geq \frac{\sqrt{10} r_{\text{comm}}}{5}
\end{cases}
\]
**Main Result**

Distance optimality guarantee

\(\Rightarrow\) Necessary and sufficient condition for distance optimality (non-stochastic)

\(\Rightarrow\) Non-asymptotic \((1 - \epsilon)\) probabilistic guarantee for \(0 < \epsilon < 1\)

\[
\begin{align*}
  n \geq & \left\lceil \frac{\left\lceil \sqrt{2} \right\rceil^2 \log \left( \frac{1}{\epsilon} \left\lceil \frac{\sqrt{2}}{r_{\text{sense}}} \right\rceil^2 \right)}{r_{\text{sense}}} \right\rceil, & r_{\text{sense}} < \frac{\sqrt{10} r_{\text{comm}}}{5} \\
  n \geq & \left\lceil \frac{\left\lceil \sqrt{5} \right\rceil^2 \log \left( \frac{1}{\epsilon} \left\lceil \frac{\sqrt{5}}{r_{\text{comm}}} \right\rceil^2 \right)}{r_{\text{comm}}} \right\rceil, & r_{\text{sense}} \geq \frac{\sqrt{10} r_{\text{comm}}}{5}
\end{align*}
\]

\(\Rightarrow\) Tight asymptotic bounds for high-probability guarantee

Performance of decentralized, hierarchical strategies

\(\Rightarrow\) Upper bound on the distance cost for arbitrary robot/target distribution
Main Result

Distance optimality guarantee
⇒ Necessary and sufficient condition for distance optimality (non-stochastic)
⇒ Non-asymptotic \((1 - \epsilon)\) probabilistic guarantee for \(0 < \epsilon < 1\)

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n \geq \begin{cases} 
\left(\frac{\sqrt{2}}{r_{\text{sense}}}\right)^2 \log \left(\frac{1}{\epsilon} \left[\frac{\sqrt{2}}{r_{\text{sense}}}\right]^2\right), & r_{\text{sense}} < \frac{\sqrt{10} r_{\text{comm}}}{5} \\
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\end{cases}
\]

⇒ Tight asymptotic bounds for high-probability guarantee

Performance of decentralized, hierarchical strategies
⇒ Upper bound on the distance cost for arbitrary robot/target distribution
⇒ \(O(1)\) asymptotic optimality guarantee under the uniform distribution
Theorem (Necessary and Sufficient Conditions for Distance Optimality).
Under sensing and communication constraints, distance optimality can be guaranteed if and only if at $t = 0$,
1. Every robot can communicate with every other robot,
2. Each target is observable by some robot.
Theorem (Necessary and Sufficient Conditions for Distance Optimality). Under sensing and communication constraints, distance optimality can be guaranteed if and only if at $t = 0$,

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Distance Optimality Guarantee

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1. Every robot can communicate with every other robot,
2. Each target is observable by some robot.
Non-Asymptotic Optimality Guarantee
Lemma. Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

$$n \geq \left[ \frac{\sqrt{5}}{r_{comm}} \right]^2 \log \left( \frac{1}{\epsilon} \left[ \frac{\sqrt{5}}{r_{comm}} \right]^2 \right).$$
Lemma. Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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Lemma. Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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Lemma. Given $r_{comm}$ and fixing $0 < \varepsilon < 1$, $G(0)$ is connected with probability at least $1 - \varepsilon$ if

$$n \geq \left\lceil \frac{\sqrt{5}}{r_{comm}} \right\rceil^2 \log \left( \frac{1}{\varepsilon} \left\lceil \frac{\sqrt{5}}{r_{comm}} \right\rceil^2 \right).$$

$$P(n_i = 0) = \left(1 - \frac{1}{m}\right)^n$$

$$\sqrt{m} = \lceil \frac{\sqrt{5}}{r_{comm}} \rceil$$
Lemma. Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

$$n \geq \left(\frac{\sqrt{5}}{r_{comm}}\right)^2 \log \left(\frac{1}{\epsilon} \left[\frac{\sqrt{5}}{r_{comm}}\right]^2\right).$$

$$P(n_i = 0) = \left(1 - \frac{1}{m}\right)^n < e^{-\frac{n}{m}}$$
Non-Asymptotic Optimality Guarantee

Lemma. Given $r_{\text{comm}}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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$$P(n_i = 0) = \left( 1 - \frac{1}{m} \right)^n < e^{-\frac{n}{m}}$$

$$P \left( \bigcup_{i=1}^{m} E(n_i = 0) \right) \leq \sum_{i=1}^{m} P(n_i = 0)$$

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Lemma. Given $r_{\text{comm}}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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$$\begin{align*}
P(n_i = 0) &= \left(1 - \frac{1}{m}\right)^n < e^{-\frac{n}{m}} \\
P\left(\bigcup_{i=1}^{m} E(n_i = 0)\right) &\leq \sum_{i=1}^{m} P(n_i = 0) < me^{-\frac{n}{m}}
\end{align*}$$
Non-Asymptotic Optimality Guarantee

**Lemma.** Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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**Non-Asymptotic Optimality Guarantee**

**Lemma.** Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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**Theorem (Random Geometric Graphs [Penrose ’97]).** For $n$ uniformly distributed nodes in the unit square, let $G(0)$ be the communication graph for a given $r_{comm}$ at $t = 0$. Then for any real number $c$, as $n \to \infty$ (i.e., $r_{comm} \to 0$),

$$P(G \text{ is connected} \mid \pi n r_{comm}^2 - \log n \leq c) = e^{-e^c}.$$  

**Theorem [Xue & Kumar ‘04].** For $n$ uniformly distributed nodes in the unit square, the network is asymptotically connected if and only if each node has $\Theta(\log n)$ neighbors.
Lemma. Given $r_{comm}$ and fixing $0 < \epsilon < 1$, $G(0)$ is connected with probability at least $1 - \epsilon$ if

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Theorem [Xue & Kumar ‘04]. For $n$ uniformly distributed nodes in the unit square, the network is asymptotically connected if and only if each node has $\Theta(\log n)$ neighbors.
Theorem (Non-Asymptotic Bounds) Fixing $0 < \varepsilon < 1$, robots can communicate with each other and all targets are observable at $t = 0$ with probability at least $1 - \varepsilon$ when

$$n \geq \begin{cases} \frac{\sqrt{2}}{r_{\text{sense}}} \log \left( \frac{1}{\varepsilon} \frac{\sqrt{2}}{r_{\text{sense}}} \right), & r_{\text{sense}} < \frac{\sqrt{10} r_{\text{comm}}}{5} \\ \frac{\sqrt{5}}{r_{\text{comm}}} \log \left( \frac{1}{\varepsilon} \frac{\sqrt{5}}{r_{\text{comm}}} \right), & r_{\text{sense}} \geq \frac{\sqrt{10} r_{\text{comm}}}{5} \end{cases}$$
Theorem (Non-Asymptotic Bounds) Fixing $0 < \epsilon < 1$, robots can communicate with each other and all targets are observable at $t = 0$ with probability at least $1 - \epsilon$ when

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$n = \Theta(-\frac{1}{r_{\text{comm}}^2} \log r_{\text{comm}})$ is **sufficient** and **necessary** for high probability asymptotic guarantee on the connectivity of $G(0)$. 
An Ideal Hierarchical Strategy
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Ideal: $r_{comm}, r_{sense}$ as large as needed
An Ideal Hierarchical Strategy

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Hierarchical: The unit square is partitioned into $m$ small squares (here, $m = 4$)
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$\frac{1}{\sqrt{m}}$
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Bounding Distance Cost at Lower Hierarchy

\[ \frac{1}{\sqrt{m}} \]
Bounding Distance Cost at Lower Hierarchy

\[
\frac{1}{\sqrt{m}}
\]
Bounding Distance Cost at Lower Hierarchy

\[ q_i \]

\[ \frac{1}{\sqrt{m}} \]
Theorem [Talagrand ‘92] Let \( X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_n\} \) be two sets sampled i. i. d. from the same arbitrary distribution on \([0,1]^2\). Then

\[
E \left[ \min_{\sigma} \sum_{i=1}^{n} |x_i - y_{\sigma(i)}| \right] \leq C \sqrt{n \log n},
\]

in which \( C \) is a universal constant.
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\[
\sum_{i=1}^{m} E[D_i] \leq C \sqrt{m} \sum_{i=1}^{m} \frac{1}{m} \sqrt{n_i \log n_i}
\]

\[
\leq C \sqrt{m} \sqrt{\frac{\sum_i n_i}{m} \log \frac{\sum_i n_i}{m}} \leq C \sqrt{n \log n}
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in which \( C \) is a universal constant.
Bounding Distance Cost at Higher Hierarchy

$q_i$
Bounding Distance Cost at Higher Hierarchy

$q_i$
Bounding Distance Cost at Higher Hierarchy

\[ P(x_j \in q_i) = P(y_j \in q_i) = p_i \]
Bounding Distance Cost at Higher Hierarchy

\[ P(x_j \in q_i) = P(y_j \in q_i) = p_i \]

\[ P(x_j \in q_i, y_j \notin q_i) = P(x_j \notin q_i, y_j \in q_i) = p_i(1 - p_i) \]
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\[ Z_j = \begin{cases} 
-1, & x_j \notin q_i, y_j \in q_i \\
1, & x_j \in q_i, y_j \notin q_i, \\
0, & \text{otherwise}
\end{cases} \]
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\[ S_i = Z_1 + \cdots + Z_n \]
Bounding Distance Cost at Higher Hierarchy

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\[ S_i = Z_1 + \cdots + Z_n \]

\[ E[S_i^2] = nE[Z_j^2] = 2np_i(1 - p_i) \]
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\[ E[|S_i|] = E\left[\sqrt{S_i^2}\right] \leq \sqrt{E[S_i^2]} \]
Bounding Distance Cost at Higher Hierarchy

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P(x_j \in q_i) = P(y_j \in q_i) = p_i
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\Rightarrow \quad E[|S_i|] \leq \sqrt{2np_i(1 - p_i)}
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Bounding Distance Cost at Higher Hierarchy

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\[ \sum_{i=1}^{m} E[|S_i|] = \sum_{i=1}^{m} \sqrt{2np_i(1 - p_i)} = m\sqrt{2n} \sum_{i=1}^{m} \frac{1}{m} \sqrt{p_i(1 - p_i)} \]
Bounding Distance Cost at Higher Hierarchy

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\[ \leq m\sqrt{2n} \sqrt{\frac{\sum_{i=1}^{m} p_i}{m} \left(1 - \frac{\sum_{i=1}^{m} p_i}{m}\right)} = \sqrt{2n(m - 1)} \]
Theorem (Performance Upper-Bound of Ideal Hierarchical Strategies) Let $D_n$ be the total distance of an ideal hierarchical strategy with $h$ hierarchies and $m_i$ regions at hierarchy $i$, then for arbitrary distribution on $[0,1]^2$,

$$E[D_n] \leq C \sqrt{n \log n} + 2\sqrt{n} \sum_{i=1}^{h-1} \sqrt{\frac{m_{i+1}}{m_i}}.$$
Bounds on Distance Optimality

**Theorem (Performance Upper-Bound of Ideal Hierarchical Strategies)** Let $D_n$ be the total distance of an ideal hierarchical strategy with $h$ hierarchies and $m_i$ regions at hierarchy $i$, then for arbitrary distribution on $[0,1]^2$, 

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**Theorem** [Ajtai et al. ‘84]. Under the uniform distribution, with high probability, $C_1 \sqrt{n \log n} \leq D^*_n \leq C_2 \sqrt{n \log n}$. 

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Theorem [Ajtai et al. ‘84]. Under the uniform distribution, with high probability, $C_1 \sqrt{n \log n} \leq D_n^* \leq C_2 \sqrt{n \log n}$.

Corollary. With uniform distribution, fixing $h$ and $\{m_i\}$, as $n \to \infty$,

$$\frac{E[D_n]}{E[D_n^*]} \to O(1).$$
Corollary. With uniform distribution, fixing $h$ and $\{m_i\}$, as $n \to \infty$,

$$\frac{E[D_n]}{E[D^*_n]} \to O(1).$$
Incorporating Arbitrary $r_{comm}$ and $r_{sense}$
Incorporating Arbitrary $r_{\text{comm}}$ and $r_{\text{sense}}$
Incorporating Arbitrary $r_{comm}$ and $r_{sense}$

Two-level decentralized hierarchical strategy

Two-level ideal hierarchical strategy

$D_n/D^*$ vs $n$ - number of robots

$m^2 = 81$
$m^2 = 256$
$m^2 = 625$
$m^2 = 1296$

$D_n/D^*$ vs $n$ - number of robots

$r_{comm} = 0.16$
$r_{comm} = 0.09$
$r_{comm} = 0.057$
$r_{comm} = 0.004$
Incorporating Arbitrary $r_{\text{comm}}$ and $r_{\text{sense}}$

- Two-level decentralized hierarchical strategy
- Arbitrary $r_{\text{sense}}$ can also be handled similarly.

Two-level ideal hierarchical strategy

Two-level decentralized hierarchical strategy
Summary of Contribution

- Guarantee on the distance optimality of the stochastic target assignment problem
  - Necessary and sufficient condition for optimality
  - Non-asymptotic probabilistic bounds
  - Asymptotically tight bounds for high-probability guarantee
- Performance of decentralized hierarchical strategies
  - General upper bounds for arbitrary distributions
  - $O(1)$ approximation algorithm for the uniform distribution
- Important takeaway: **locally** optimal behavior leads to near *globally* optimal behavior