Outline

Sets, set operations, Venn diagrams
Cardinality, open sets, closed sets, continuous functions
Group theory concepts
Topological space concepts
Homeomorphism
Manifolds
A set is a collection of elements. Examples:

- \{1, a, cup, π\}
- All natural numbers, \(\mathbb{N}\)
- \(n\)-dimensional Euclidean spaces, \(\mathbb{R}^n\)

Set operations

- Union: \(A \cup B = \{x \mid x \in A \lor x \in B\}\)
- Intersection: \(A \cap B = \{x \mid x \in A \land x \in B\}\)
- Complement: \(\overline{A} = \{x \mid x \in U \land x \notin A\}\)
- Difference: \(A - B = \{x \mid x \in A \land x \notin B\}\)
- Symmetric difference: \(A \ominus B = A \cup B - A \cap B\)

Venn diagram
More Sets and Functions

Subset (⊂): \( B \subset A \iff \forall x \in B, x \in A \)

Superset (⊃): \( A \supset B \iff B \subset A \)

Powerset: \( \mathcal{P}(S) = \{A \mid A \subset S\} \), example:
\[
\Rightarrow S = \{1, 2\}
\]
\[
\Rightarrow \mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}
\]

Functions

\[
\Rightarrow \text{To fully specify a function we write } f : X \rightarrow Y, x \mapsto f(x)
\]
\[
\Rightarrow \text{For } f \text{ to be a function, } \forall x \in X, f(x) \in Y \text{ must be uniquely defined}
\]
\[
\Rightarrow \text{Ex: for } f(x) = x^2, \text{ we write } f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2
\]
\[
\Rightarrow \text{A function is surjective if } f(X) = Y. \text{ Ex: } f : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}, x \mapsto x^2
\]
\[
\Rightarrow \text{A function is injective if } \forall x_1 \neq x_2, f(x_1) \neq f(x_2)
\]
\[
\Rightarrow \text{Ex: } f : \mathbb{N} \rightarrow \mathbb{N}, x \mapsto x^2
\]
\[
\Rightarrow \text{A function is bijective if it is both surjective and injective}
\]
\[
\Rightarrow \text{Ex: } f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^3
\]
Cardinality of Sets

Cardinality: essentially the “size” of a set

⇒ |∅| = 0
⇒ |{1, 2}| = 2
⇒ |\mathcal{P}({1,2})| = 4
⇒ |\mathcal{P}(S)| = 2^{|S|}
⇒ |\mathbb{N}| = \aleph_0 - the “smallest” infinite cardinal number
⇒ |\mathbb{R}| = \aleph_1

Measuring the relative cardinality of sets

⇒ |A| \leq |B| if there exists an injective function f: A → B
⇒ If |A| \leq |B| and |B| \leq |A|, then |A| = |B|
  ⇒ This means there is a bijective function between A and B

⇒ Cardinality of rational numbers?
  ⇒ Same as the set of natural numbers

⇒ Cardinality of real numbers?
  ⇒ Uncountably infinite (Cantor’s diagonalization argument)
Open Sets, Closed Set, Boundary on $\mathbb{R}^n$

In Euclidean spaces, by convention, a set $X$ is open if for all $x \in X$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X$.

- Ex: $\mathbb{R}$ is open, $\forall a, b \in \mathbb{R}, a < b, (a, b)$ is open
- Ex: The set $\{(x, y) | x^2 + y^2 < 1\} \subset \mathbb{R}^2$ is open
- The union of any number of open set is open

A set is closed if its complement is open

- A set may be neither open nor closed, e.g., $(a, b]$  

The closure of a set is the set plus all its limit points

- The closure of a set is closed, e.g. $Cl(S)$ is always closed
  - Also, $Cl(S) = Cl(Cl(S))$

The interior of a set $S$, denoted $S^o$, is the union of all open sets in $S$

The boundary of a set $S$, denoted $\partial S$, is $Cl(S) - S^o$. 
Continuous Functions

A function \( f: \mathbb{R} \to \mathbb{R} \) is continuous around \( x_0 \in \mathbb{R} \) if \( \forall \varepsilon > 0, \exists \delta > 0 \), s.t. \( \forall x \in B(x_0, \delta), f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \).

\( \Rightarrow \) Readily generalize to \( f: \mathbb{R}^n \to \mathbb{R} \)

\( \Rightarrow \) Alternative definition based on open sets: a function is continuous if the preimages of open sets are open.
Group Theory Concepts

A set $G$ together with an binary operation $\cdot$ is a group if the following group axioms are satisfied

$\Rightarrow$ **Closed:** $\forall a, b \in G, a \cdot b \in G$

$\Rightarrow$ **Associative:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

$\Rightarrow$ **Identity:** $\exists e \in G, \forall a \in G, a \cdot e = e \cdot a = a$

$\Rightarrow$ **Inverse:** $\forall a \in G, \exists b \in G$ s.t. $a \cdot b = b \cdot a = e$

There is a unique identity

$\Rightarrow$ Suppose $e_1, e_2 \in G$ are both identities, then $e_1 = e_1 \cdot e_2 = e_2$

Inverses are unique

$\Rightarrow$ For $a \in G$, suppose $b, c \in G$ are its inverses, then $b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c$

Some examples

$\Rightarrow$ The set of natural numbers under addition

$\Rightarrow$ The set of positive rational numbers under multiplication
Why Groups?

It a mathematical field full of sad (but curious) stories!

⇒ Niels Henrik Abel (Norwegian, 1802-1829)
   ⇒ Invented group theory!
   ⇒ Proved no explicit algebraic for quintic polynomials
   ⇒ And many other fundamental contributions...
   ⇒ Very unlucky!
      ⇒ Sent group theory paper to Gauss, Gauss tossed it into garbage...
      ⇒ Sent another seminar paper to Cauchy, Cauchy misplaced it...
      ⇒ Died at the age of 26!
      ⇒ Then he got a letter appointing him Professor at University of Berlin
   ⇒ Abel Prize is basically the Nobel Prize in math

⇒ Évariste Galois (French, 1811-1832)
   ⇒ Also invented group theory (independently, no internet then)
   ⇒ Galois theory (more general than Abel’s work)
   ⇒ Also many many other important work...
   ⇒ But this guy was very passionate
   ⇒ Political activist, went to prison
   ⇒ Then chose to dual with an army officer and died...
   ⇒ 20 years old!
Why Groups? Seriously...

Many types of discrete and continuous spaces are also groups!

- Ex: The Rubik’s cube
  - It’s a planning problem!
- Ex: \( \mathbb{R} \) under addition
- Ex: The unit circle under rotation

⇒ Can also do this using matrix

\[
\begin{bmatrix}
\cos(\alpha + \beta) & -\sin(\alpha + \beta) \\
\sin(\alpha + \beta) & \cos(\alpha + \beta)
\end{bmatrix} =
\begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{bmatrix}
\]

⇒ More on this when we do transformations
Topological Space

A set $X$ and a collection $\Gamma$ of subsets of $X$ form a topological space if

- $\emptyset \in \Gamma$ and $X \in \Gamma$
- Arbitrary union of elements of $\Gamma$ is again in $\Gamma$
- Finite intersection of elements of $\Gamma$ is again in $\Gamma$

Note: here, “open sets” are defined differently from earlier

- E.g., point set topologies (from Wikipedia)

A set $A$ is closed if $X - A$ is open
Topological Spaces on $\mathbb{R}$

The standard topology on $\mathbb{R}$ is the one with $\{(a, b)\} \cup \{\mathbb{R}\} \subset \Gamma$ for all $a \leq b$. This is similar to what have done before

$\Rightarrow$ Is $[0, 1]$ open or closed?
$\Rightarrow$ Closed, because $(-\infty, 0) \cup (1, \infty)$ is open

$\Rightarrow$ What about $\bigcup_{i=1}^{\infty} \left( i, i + \frac{1}{i} \right)$?

Alternatively, we can have $\Gamma = \{\emptyset, \mathbb{R}\}$

$\Rightarrow$ This is the **trivial topology** on $\mathbb{R}$

Or, we can have $\Gamma = \{ (-n, n) | n \in \mathbb{R} \} \cup \{\emptyset, \mathbb{R}\}$

Similar topologies can be defined for $\mathbb{R}^n$
Homeomorphism (I)

Why study topology?

⇒ One of the use is that it helps us to classify spaces
⇒ Which of the following spaces are similar?

⇒ The first and the third are both "one dimensional"
⇒ What about those?

⇒ All similar to the circle

Homeomorphism: two spaces $X$ and $Y$ are homeomorphic if there is a continuous function $F: X \rightarrow Y$ that is bijective
Homeomorphism (II)

We can build a bijection $f: X \rightarrow Y$ by “sliding” from one end to another on both lines.

What about
Homeomorphism (III)

One can build a series of homeomorphism to deform between two objects, that is,

\[ F_t, \ t \in [0, 1], \] is a bijective continuous function with domain \( X \)

\[ F_0(X) = X, \ F_1(X) = Y \]

This is also known as a deformation

Classic example: coffee mug and donut